Teleparallel Killing Vectors of the Einstein Universe

M. Sharif *and M. Jamil Amir †
Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus, Lahore-54590, Pakistan.

Abstract

In this short paper we establish the definition of the Lie derivative of a second rank tensor in the context of teleparallel theory of gravity and also extend it for a general tensor of rank $p+q$. This definition is then used to find Killing vectors of the Einstein universe. It turns out that Killing vectors of the Einstein universe in the teleparallel theory are the same as in General Relativity.

Keywords: Teleparallel Theory, Killing Vectors.

1 Introduction

In General Relativity (GR), the importance of symmetry is quite clear. The symmetry restrictions are very much helpful to find the solution of the Einstein field equations (EFEs). Much attention has been given to study the different kinds of symmetries during the last three decades. As a pioneer, Petrov [1] solved the Killing equations for four-dimensional spaces. Bokhari and Qadir [2-3] were able to achieve a complete classification of static spherically symmetric spacetimes. Later, Qadir and Ziad extended this work for a complete classification of all spherically symmetric spacetimes [4-5] by removing the condition of staticity. The solutions of the EFEs, corresponding

*msharif@math.pu.edu.pk
†mjamil.dgk@gmail.com

to different symmetries possessed by the metric tensor, have further been classified according to their properties and groups of motion admitted by them [6].

Katzin et al. [7-8] studied the curvature and Ricci collineations (RCs) in the context of the related particle and field conservation laws. A detail investigation of the spacetimes and their geometrical symmetries like Killing vectors (KVs), curvature collineations and RCs was made by different authors [9-13]. Using Lie algebra approach, Carot et al. [14] discussed the physical properties of the spacetimes called matter collineations (MCs). Camci and Sharif [15] worked out the MCs of homogeneous Gödel-type metrics. One of the authors (MS) [16-17] found the MCs of the spherically symmetric spacetimes (both static and non-static). Later, this work was extended to classify static plane and cylindrically symmetric spacetimes according to their MCs by the same author [18-19].

Teleparallel theory of gravity (TPG) is an alternative theory of gravity which corresponds to a gauge theory of translation group [20] based on Weitzenböck geometry [21]. This theory is characterized by the vanishing of curvature identically while the torsion is taken to be non-zero. In TPG, gravitation is attributed to torsion which plays a role of force [22-25]. In GR, gravitation geometrizes the underlying spacetime. The translational gauge potentials appear as a non-trivial part of the tetrad field and induce a teleparallel (TP) structure on spacetime which is directly related to the presence of a gravitational field. In some other theories [22-27], torsion is only relevant when spins are important. This point of view indicates that torsion might represent additional degrees of freedom as compared to curvature

There is a large body of literature available [28-30 and references there in] about the study of TP versions of the exact solutions of GR. Pereira, et al. [30] obtained the TP versions of the Schwarzschild and the stationary axisymmetric Kerr solutions of GR. They proved that the axial-vector torsion plays the role of the gravitomagnetic component of the gravitational field in the case of slow rotation and weak field approximations. In recent papers [31-33], we have found the TP versions of the Friedmann models, Lewis-Papapetrou spacetimes and stationary axisymmetric solutions. The energy-momentum distribution of these versions have also been discussed.

This paper is devoted to look at the symmetry of a metric tensor in the context of TPG. For this purpose, we define the TP version of the Lie derivative which gives Killing equations in TPG. This has been used to find the TP KVs of the Einstein universe giving the comparison of KVs with GR.

The paper is organized as follows. Section 2 contains an overview of the TP theory. In section 3, we define the Lie derivative in the framework of TPG and the Killing equations. Section 4 is devoted to explore the TP KVs of the Einstein universe. The last section concludes the results obtained.

2 An Overview of the Teleparallel Theory

The TP theory is based on the Weitzenböck connection given as [25]

$$\Gamma^{\theta}{}_{\mu\nu} = h_a{}^{\theta}\partial_\nu h^a{}_\mu, \tag{1}$$

where $h_a{}^{\nu}$ is a non-trivial tetrad. Its inverse field is denoted by $h^a{}_\mu$ and satisfies the relations

$$h^a{}_\mu h_a{}^\nu = \delta_\mu{}^\nu, \quad h^a{}_\mu h_b{}^\mu = \delta^a{}_b. \tag{2}$$

In this paper, the Latin alphabet $(a, b, c, ... = 0, 1, 2, 3)$ will be used to denote the tangent space indices and the Greek alphabet $(\mu, \nu, \rho, ... = 0, 1, 2, 3)$ to denote the spacetime indices. The Riemannian metric in TP theory arises as a by product [25] of the tetrad field given by

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu, \tag{3}$$

where η_{ab} is the Minkowski metric, i.e., $\eta_{ab} = diag(+1, -1, -1, -1)$. For the Weitzenböck spacetime, the torsion is defined as

$$T^{\theta}{}_{\mu\nu} = \Gamma^{\theta}{}_{\nu\mu} - \Gamma^{\theta}{}_{\mu\nu} \tag{4}$$

which is antisymmetric w.r.t. its last two indices. Due to the requirement of absolute parallelism, the curvature of the Weitzenböck connection vanishes identically. The Weitzenböck connection also satisfies the relation

$$\Gamma^{0\theta}{}_{\mu\nu} = \Gamma^{\theta}{}_{\mu\nu} - K^{\theta}{}_{\mu\nu}, \tag{5}$$

where

$$K^{\theta}{}_{\mu\nu} = \frac{1}{2}(T_\mu{}^{\theta}{}_\nu + T_\nu{}^{\theta}{}_\mu - T^{\theta}{}_{\mu\nu}) \tag{6}$$

is the **contortion tensor** and $\Gamma^{0\theta}{}_{\mu\nu}$ are the Christoffel symbols in GR.

3 TP Version of the Lie Derivative and the Killing Equations

For a scalar, the TP Lie derivative will act as the directional derivative which is given as

$$\mathcal{L}^T{}_\xi \phi = \xi^\rho \nabla_\rho \phi = \xi^\rho \frac{\partial \phi}{\partial \xi^\rho}, \tag{7}$$

where $\mathcal{L}^T{}_\xi$ is used to denote the TP Lie derivative along the vector field ξ. The TP Lie derivative of a covariant tensor of rank 2 along a vector field ξ is defined through the covariant derivatives of that tensor and vector field as

$$(\mathcal{L}^T{}_\xi \mathbf{A})_{\mu\nu} = \xi^\rho \nabla_\rho A_{\mu\nu} + (\nabla_\mu \xi^\rho) A_{\rho\nu} + (\nabla_\nu \xi^\rho) A_{\mu\rho}, \tag{8}$$

where ∇_ρ stands for the TP covariant derivative and is defined [25] as

$$\nabla_\rho A_{\mu\nu} = A_{\mu\nu,\rho} - \Gamma^\theta_{\rho\nu} A_{\mu\theta} - \Gamma^\theta_{\mu\rho} A_{\nu\theta}, \tag{9}$$

$\Gamma^\theta_{\rho\nu}$ are the Weitzenböck connection as given by Eq.(1). In view of Eq.(9), Eq.(8) takes the form

$$\begin{aligned}(\mathcal{L}^T{}_\xi A)_{\mu\nu} &= \xi^\rho (A_{\mu\nu,\rho} - \Gamma^\theta{}_{\rho\nu} A_{\mu\theta} - \Gamma^\theta{}_{\mu\rho} A_{\nu\theta}) + A_{\rho\nu}(\xi^\rho{}_{,\mu} + \Gamma^\rho{}_{\theta\mu}\xi^\theta) \\ &+ A_{\mu\rho}(\xi^\rho{}_{,\nu} + \Gamma^\rho{}_{\theta\nu}\xi^\theta).\end{aligned} \tag{10}$$

After some simple calculations and using Eq.(4), we get

$$(\mathcal{L}^T{}_\xi A)_{\mu\nu} = A_{\mu\nu,\rho}\xi^\rho + A_{\rho\nu}\xi^\rho{}_{,\mu} + A_{\mu\rho}\xi^\rho{}_{,\nu} + \xi^\rho (A_{\theta\nu} T^\theta{}_{\mu\rho} + A_{\mu\theta} T^\theta{}_{\nu\rho}). \tag{11}$$

Similarly, the TP Lie derivative of a contravariant tensor of rank 2 can be written as

$$(\mathcal{L}^T{}_\xi A)^{\mu\nu} = A^{\mu\nu}{}_{,\rho}\xi^\rho - A^{\rho\nu}\xi^\mu{}_{,\rho} + A^{\mu\rho}\xi^\nu{}_{,\rho} - \xi^\rho (A^{\theta\nu} T^\mu{}_{\theta\rho} + A^{\mu\theta} T^\nu{}_{\theta\rho}). \tag{12}$$

Following the same procedure, we can extend this definition for a mixed tensor of rank $p+q$ as

$$\begin{aligned}(\mathcal{L}^T{}_\xi A)^{\rho\ldots\sigma}{}_{\mu\ldots\nu} &= \xi^\alpha A^{\rho\ldots\sigma}{}_{\mu\ldots\nu,\alpha} + A^{\rho\ldots\sigma}{}_{\alpha\ldots\nu}\xi^\alpha_{,\mu} + \ldots + A^{\rho\ldots\sigma}{}_{\mu\ldots\alpha}\xi^\alpha_{,\nu} \\ &- A^{\alpha\ldots\sigma}{}_{\mu\ldots\nu}\xi^\rho_{,\alpha} - \ldots - A^{\rho\ldots\alpha}{}_{\mu\ldots\nu}\xi^\sigma_{,\alpha} \\ &+ \xi^\alpha (A^{\rho\ldots\sigma}{}_{\beta\ldots\nu} T^\beta{}_{\mu\alpha} + \ldots + A^{\rho\ldots\sigma}{}_{\mu\ldots\beta} T^\beta{}_{\nu\alpha} \\ &- A^{\beta\ldots\sigma}{}_{\mu\ldots\nu} T^\rho{}_{\beta\alpha} - \ldots - A^{\rho\ldots\beta}{}_{\mu\ldots\nu} T^\sigma{}_{\beta\alpha}),\end{aligned} \tag{13}$$

where $|\{\rho...\sigma\}| = p$ and $|\{\mu...\nu\}| = q$. Now, we can define the TP Killing equations as

$$(\mathcal{L}^T{}_\xi g)_{\mu\nu} = 0. \tag{14}$$

Using Eq.(11), Eq.(14) becomes

$$(\mathcal{L}^T{}_\xi g)_{\mu\nu} = g_{\mu\nu,\rho}\xi^\rho + g_{\rho\nu}\xi^\rho{}_{,\mu} + g_{\mu\rho}\xi^\rho{}_{,\nu} + \xi^\rho(g_{\theta\nu}T^\theta{}_{\mu\rho} + g_{\mu\theta}T^\theta{}_{\nu\rho}). \tag{15}$$

4 TP Killing Vectors of the Einstein Universe

The metric representing the Einstein universe is given as follows

$$ds^2 = dt^2 - \frac{1}{A^2(r)}dr^2 - d\theta^2 - r^2 sin^2\theta d\phi^2, \tag{16}$$

where $A(r) = \sqrt{1 - \frac{r^2}{R^2}}$ and R is constant. Using the procedure adopted in the papers [30-33], the tetrad components of the above metric can be written as

$$h^a{}_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{A}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \frac{1}{A}\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \frac{1}{A}\cos\theta & -r\sin\theta & 0 \end{bmatrix} \tag{17}$$

with its inverse

$$h^a{}_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & A\sin\theta\cos\phi & \frac{1}{r}\cos\theta\cos\phi & -\frac{\sin\phi}{r\sin\theta} \\ 0 & A\sin\theta\sin\phi & \frac{1}{r}\cos\theta\sin\phi & \frac{\cos\phi}{r\sin\theta} \\ 0 & A\cos\theta & -\frac{1}{r}\sin\theta & 0 \end{bmatrix}. \tag{18}$$

When we make use of Eqs.(17) and (18) in Eq.(1), we obtain the following non-vanishing components of the Weitzenböck connection

$$\begin{aligned} \Gamma^1{}_{11} &= -\frac{A'}{A}, \quad \Gamma^1{}_{22} = -rA, \quad \Gamma^1{}_{33} = -rA\sin^2\theta, \\ \Gamma^2{}_{12} &= \frac{1}{rA} = \Gamma^3{}_{13}, \quad \Gamma^2{}_{21} = \frac{1}{r} = \Gamma^3{}_{31}, \\ \Gamma^2{}_{33} &= -\sin\theta\cos\theta, \quad \Gamma^3{}_{23} = \cot\theta = \Gamma^3{}_{32}. \end{aligned} \tag{19}$$

The corresponding non-vanishing components of the torsion tensor are

$$T^2{}_{12} = \frac{1}{r}(1 - \frac{1}{A}) = T^3{}_{13}. \tag{20}$$

Making use of Eq.(20) in (15), we get the TP Killing equations in the expanded from as

$$\xi^1 = AB(t, \theta, \phi), \tag{21}$$
$$\xi^1 + rA\xi^2_{,2} = 0, \tag{22}$$
$$\frac{1}{A}\xi^1 + r\cot\theta\xi^2 + r\xi^3_{,3} = 0, \tag{23}$$
$$\xi^1_{,0} - A^2\xi^0_{,1} = 0, \tag{24}$$
$$\xi^0_{,2} - r^2\xi^2_{,0} = 0, \tag{25}$$
$$\xi^0_{,3} - r^2\sin^2\theta\xi^3_{,0} = 0, \tag{26}$$
$$\xi^2_{,1} + \frac{1}{A^2r^2}\xi^1_{,2} + \frac{1}{r}(1 - \frac{1}{A})\xi^2 = 0, \tag{27}$$
$$\xi^3_{,1} + \frac{1}{A^2r^2\sin^2\theta}\xi^1_{,3} + \frac{1}{r}(1 - \frac{1}{A})\xi^3 = 0, \tag{28}$$
$$\xi^2_{,3} + \sin^2\theta\xi^3_{,2} = 0, \tag{29}$$
$$\xi^0 = N(r, \theta, \phi). \tag{30}$$

Solving Eqs.(21), (22) and (27) simultaneously, it follows that

$$\xi^1 = A\{B_1(t, \phi)\cos\theta + B_2(t, \phi)\sin\theta\}, \tag{31}$$
$$\xi^2 = -\frac{1}{r}\{B_1(t, \phi)\sin\theta - B_2(t, \phi)\cos\theta\} + F(r)C(t, \phi), \tag{32}$$

where $F(r) = \frac{1}{r^2}(R - \sqrt{R^2 - r^2})$. Substituting the value of ξ^2 in Eq.(29), we get

$$\begin{aligned} \xi^3 &= \frac{1}{r}\{B_{1,3}(t, \phi)\ln|\csc\theta - \cot\theta| + B_{2,3}(t, \phi)\csc\theta\} \\ &+ F(r)C_{,3}(t, \phi)\cot\theta + D(t, r, \phi), \end{aligned} \tag{33}$$

Using these values in the remaining equations, it follows that

$$\begin{aligned} \xi^0 &= c_0, \\ \xi^1 &= A\{c_1\cos\theta + (c_2\cos\phi + c_3\sin\phi)\sin\theta\}, \end{aligned}$$

$$
\begin{aligned}
\xi^2 &= -\frac{1}{r}\{c_1 \sin\theta - (c_2\cos\phi + c_3\sin\phi)\cos\theta\} \\
&+ F(r)(c_4\cos\phi + c_5\sin\phi), \\
\xi^3 &= \frac{1}{r}\{(c_3\cos\phi - c_2\sin\phi)\csc\theta\} + c_6F(r) \\
&- F(r)\cot\theta(c_4\sin\phi - c_5\cos\phi). \qquad (34)
\end{aligned}
$$

This gives the following 7 KVs of the Einstein universe in the context of TPG

$$
\begin{aligned}
\xi_{(1)} &= \frac{\partial}{\partial t}, \\
\xi_{(2)} &= F(r)\frac{\partial}{\partial\phi} \\
\xi_{(3)} &= F(r)(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}), \\
\xi_{(4)} &= F(r)(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}), \\
\xi_{(5)} &= A\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}, \\
\xi_{(6)} &= A\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\phi\cos\theta\frac{\partial}{\partial\theta} - \frac{1}{r}\sin\phi\csc\theta\frac{\partial}{\partial\phi}, \\
\xi_{(7)} &= A\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\sin\phi\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{r}\cos\phi\csc\theta\frac{\partial}{\partial\phi}. \qquad (35)
\end{aligned}
$$

5 Summary and Discussion

In this paper, we have defined the Lie derivative on the space with torsion. This definition is then applied to find the TP Killing vectors of the Einstein universe. It is shown that there arise 7 TP KVs which are the same in numbers as found in the context of GR [2-3]. The comparison shows that $\xi_{(1)}$ is the same in both the theories while $\xi_{(2)}$, $\xi_{(3)}$ and $\xi_{(4)}$ in TPG are multiple of the corresponding KVs in GR by $F(r)$. For $F(r) = 1$, that is, for $r = \pm\sqrt{2R-1}$, the first four KVs reduce to the basic four KVs of the spherical symmetry and coincide with those in GR. This implies that, in torsion space, the spherical symmetry can be recovered for a particular choice of $r = \pm\sqrt{2R-1}$. The remaining three TP KVs are different from the KVs found in the context of GR. This difference occurs due to the non-vanishing components of the torsion tensor which involves in Eq.(15) when $\mu = \nu$. The

extension of this work to the other spacetimes is under investigation which may help to make a conjecture about the relationship between the KVs in GR and TP.

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